Tutorial 5 : Selected problems of Assignment 5

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Announcement ① HW1 - HW4 are marked and are ready for pick up.

② Extra office hour for Midterm: 16 Oct (Wed): 10:30-12:00

Notation $(C[a,b], \|\cdot\|_1$ (resp. $\|\cdot\|_\infty$)) normed space of continuous functions

endowed with L^1-norm (resp. sup-norm)

where
- $C[a,b] = \{ f:[a,b] \to \mathbb{R} \mid f:$ continuous $\}$
- $\|f\|_1 := \int_a^b |f|$
- $\|f\|_\infty := \sup_{x \in [a,b]} \{ |f(x)| \}$

Q1) (Ex.5, Q3) Define $\Phi: C[a,b] \to \mathbb{R}$ as $\Phi(f) = \int_a^b \sqrt{1+f^2(x)}\, dx$.

Show that Φ is continuous with respect to $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

Sol) Case for $(C[a,b], \|\cdot\|_1)$: Define an auxillary function $h: \mathbb{R} \to \mathbb{R}$ by

$h(y) = \sqrt{1+y^2}$; Then $h'(y) = \frac{1}{2\sqrt{1+y^2}} \cdot 2y = \frac{y}{\sqrt{1+y^2}} \leq 1$, for all $y \in \mathbb{R}$.

Showing Φ is Lipschitz continuous : For any $f, g \in C[a,b]$,

$$|\Phi(f) - \Phi(g)| = \left| \int_a^b \left(\sqrt{1+f^2(x)} - \sqrt{1+g^2(x)}\right) dx \right| = \left| \int_a^b \left(h(f(x)) - h(g(x)) \right) dx \right|$$

$$\leq \int_a^b |h(f(x)) - h(g(x))|\, dx \leq \int_a^b |h'(\xi(x))| \, |f(x) - g(x)|\, dx \quad \text{(By Mean Value Theorem)}$$

$$\leq \int_a^b |f-g|\, dx = \|f-g\|_1$$

$\therefore$ Φ is Lipschitz continuous, and hence continuous.

Case for $(C[a,b], \|\cdot\|_\infty)$: Recall that $\|\cdot\|_\infty$ is stronger than $\|\cdot\|_1$:

$\|F\|_1 = \int_a^b |F(x)|\, dx \leq (b-a)\|F\|_\infty$, for all $F \in C[a,b]$.

$\therefore$ For any $f, g \in C[a,b]$, $|\Phi(f) - \Phi(g)| \leq \|f-g\|_1 \leq (b-a)\|f-g\|_\infty$

$\therefore$ Φ is also (Lipschitz) continuous.

Q2) (Ex. 5, Q4) Fix $x_0 \in [a,b]$. Define $\Psi : C[a,b] \to \mathbb{R}$ as $\Psi(f) = f(x_0)$.

Show that Ψ is continuous with respect to $\|\cdot\|_\infty$ but not for $\|\cdot\|_1$.

Sol) Case for $(C[a,b], \|\cdot\|_\infty)$: Showing Ψ is Lipschitz continuous:

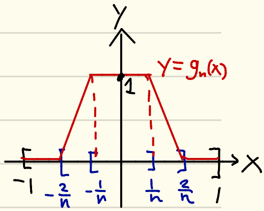
For any $f, g \in C[a,b]$, $|\Psi(f) - \Psi(g)| = |f(x_0) - g(x_0)| \leq \|f-g\|_\infty$

Case for $(C[a,b], \|\cdot\|_1)$: Constructing $\{f_n\} \subseteq C[a,b]$ such that

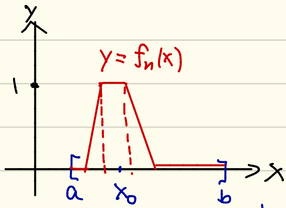
$\Psi(f_n) = 1$ for all $n \in \mathbb{N}$ but $\lim_{n\to\infty} \|f_n\|_1 = 0$: Assuming $x_0 \neq a, b$

Define $g_n : [-1,1] \to \mathbb{R}$ as

$$g_n(x) = \begin{cases} 0, & x \leq -\frac{2}{n} \\ nx+2, & -\frac{2}{n} \leq x \leq -\frac{1}{n} \\ 1, & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ -nx+2 & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & x \geq \frac{2}{n} \end{cases}$$



and $$f_n(x) := \begin{cases} g_n\left(\frac{x-x_0}{x_0-a}\right), & a \leq x \leq x_0 \\ g_n\left(\frac{x-x_0}{b-x_0}\right), & x_0 \leq x \leq b \end{cases}$$



Then for all $n \in \mathbb{N}$, $\Psi(f_n) = f_n(x_0) = g_n(0) = 1$; $\|f_n\|_1 = \int_a^b |f_n|$

$= \int_a^{x_0} |f_n| + \int_{x_0}^b |f_n| = (x_0 - a)\int_{-1}^0 |g_n| + (b-x_0)\int_0^1 |g_n| = (x_0-a)\frac{3}{2n} + (b-x_0)\frac{3}{2n} = (b-a)\frac{3}{2n}$

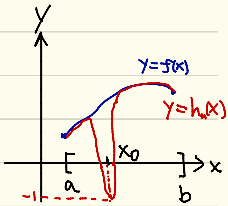
$\therefore \lim_n \|f_n\|_1 = 0$

For $x_0 = a$ or b, use the "left half" or "right half" of g_n instead.

Q3) (Ex.5, Q7) Let $P := \{ f \in C[a,b] \mid f(x) > 0, \text{ for all } x \in [a,b] \}$

Show that $P \subseteq (C[a,b], \|\cdot\|_1)$ is NOT open.

Sol) It suffices to show that given any $f \in P$, $\varepsilon > 0$, there exists $h \notin P$ such that $\|f-h\|_1 < \varepsilon$



Fix $x_0 \neq a, b$, $f_n : [a,b] \to \mathbb{R}$ as in Q2,

$h_n : [a,b] \to \mathbb{R}$ by $h_n(x) = f(x) - (f(x_0)+1) f_n(x)$

then for any $n \in \mathbb{N}$, $h_n(x_0) = f(x_0) - (f(x_0)+1) \cdot 1 = -1 < 0$. $\therefore h_n \notin P$

Also, $\|f - h_n\|_1 = \int_a^b | f(x) - (f(x) - (f(x_0)+1) f_n(x)) | dx$

$$= |f(x_0)+1| \int_a^b |f_n| = |f(x_0)+1| (b-a) \frac{3}{2n}$$

$\therefore$ Define $h = h_N$, where $N \in \mathbb{N}$ satisfies $|f(x_0)+1| (b-a) \frac{3}{2N} < \varepsilon$.

then $h \notin P$ and $\|f-h\|_1 < \varepsilon$.